Casimir - Polder force density between an atom and a conducting wall

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In this paper we calculate the Casimir - Polder force density (force per unit area acting on the elements of the surface) on a metallic plate placed in front of a neutral atom. To obtain the force density we use the quantum operator associated to the electromagnetic stress tensor. We explicitly show that the integral of this force density over the plate reproduces the total force acting on the plate. This result shows that, although the force is obtained as a sum of surface element - atom contributions, the stress - tensor method includes also nonadditive components of Casimir - Polder forces in the evaluation of the force acting on a macroscopic object.

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I. INTRODUCTION

A surprising prediction of quantum electrodynamics is the existence of forces of electromagnetic nature between pairs of metallic uncharged objects. This kind of effects were first predicted theoretically by H.B.G. Casimir and D. Polder in 1948. In two different articles (the former of Casimir alone [1], the latter of both [2]), the existence of forces between two flat parallel neutral metallic plates, the so called *Casimir effect*, between a neutral atom and a wall and between two neutral atoms is predicted. The origin of these forces is commonly attributed to the properties of the vacuum state of the electromagnetic field, that is the state of minimum energy, and in particular to the fact that the vacuum energy depends on the boundary conditions characterizing the system [3, 4]. Although these forces are very tiny, they have been measured with remarkable precision. The first successful precision experiments were on the wall - sphere system [5, 6]. Next, the force between a neutral atom and a wall was observed [7, 8, 9], as well as between two metallic neutral parallel plates [10, 11]. For the atom - atom Casimir - Polder force, only indirect evidences of their existence have been yet obtained.

In this paper we consider a neutral atom in front of a metallic, perfectly conducting plate. The force on the atom in this configuration has been calculated by Casimir and Polder [2]. We focus our attention on the plate and in Section II we calculate, using the quantum electromagnetic stress tensor, the force acting upon each surface element of the plate. Then, in Section III we show that the integral of this force density over the surface of the plate equals the opposite of the force experienced by the atom. This conducts us to a concluding enquiry on the connection between the stress - tensor method and the well - known nonadditivity of Casimir - Polder forces. We conclude that the stress - tensor method has the great advantage of automatically including many - body contri-

butions of the Casimir - Polder forces.

II. THE FORCE DENSITY ON THE WALL

The Casimir - Polder interaction between a neutral atom and a metallic uncharged wall yields an attractive atom - wall force. The value of this force was originally calculated using second order perturbation theory and the minimal coupling Hamiltonian, obtaining

$$F_A(d) = -\frac{3\hbar c\alpha}{2\pi d^5} \tag{1}$$

where d is the atom - wall distance, α is the static polarizability of the atom and the minus sign indicates that the force is attractive [2]. The expression (1) is valid in the so - called *far zone* defined by $d >> c/\omega_0$, ω_0 being a typical atomic frequency.

We now focus our attention on the wall. Being the wall an extended object, it makes sense to ask ourselves which is the force acting on each surface element of the metallic plate. To answer this question we use the quantum operator associated to the classical electromagnetic stress tensor. This method was widely used by Barton in his works about fluctuations of Casimir - Polder forces [12, 13]. In classical electrodynamics, the stress tensor is defined as

$$S_{ij}(\mathbf{r}) = \frac{1}{4\pi} \Big[E_i(\mathbf{r})E_j(\mathbf{r}) + B_i(\mathbf{r})B_j(\mathbf{r}) - \frac{1}{2}\delta_{ij}\Big(E^2(\mathbf{r}) + B^2(\mathbf{r})\Big)\Big] \tag{2}$$

where E_i and B_i are components of the electric and magnetic field, i and j assuming the values $1, 2, 3$ corresponding, respectively, to x, y and z. As it is well known [14], this tensor permits to calculate the force on a volume V. The i-th component of this force is given by

$$F_i = \int_V dV\, f_i = \oint_S \sum_{j=1}^3 dA_j\, S_{ij} \tag{3}$$

where the last integral is extended over the surface S enclosing the volume V. If we assume the wall to be

located at $z = L$ and to have an infinitesimal perpendicular extension dz, the z component of the force on an infinitesimal parallelepiped $dxdydz$ representing the surface element centered in the point (x, y) on the plate is given by

$$\sigma(x, y) = S_{zz}(x, y, L + dz) - S_{zz}(x, y, L) \qquad (4)$$

Equations (2) and (4) are classical. To obtain the quantum operator associated to the stress tensor we simply replace the components E_i and B_i of the electric and magnetic fields with the corresponding quantum operators. In the Schrödinger representation, we have

$$\mathbf{E}(\mathbf{r}) = i \sum_{\mathbf{k}j} \sqrt{\frac{2\pi\hbar\omega_k}{V}} (a_{\mathbf{k}j} - a_{\mathbf{k}j}^\dagger)\mathbf{f}(\mathbf{k}j, \mathbf{r}) \qquad (5)$$

$$\mathbf{B}(\mathbf{r}) = \sum_{\mathbf{k}j} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} (a_{\mathbf{k}j} + a_{\mathbf{k}j}^\dagger)[\nabla \times \mathbf{f}(\mathbf{k}j, \mathbf{r})] \qquad (6)$$

where the functions $\mathbf{f}(\mathbf{k}j, \mathbf{r})$ (assumed real) are the field modes corresponding to the boundary conditions characterizing the system. In our case, the presence of a metallic surface located in $z = L$ can be taken into account by considering two metallic boxes (the former on the left, the latter on the right of the wall) having in common one side on this plane. At the end of the calculations, we send to infinity the length of the three edges of the two cavities. For example, the cavity on the right of the plane is the parallelepiped

$$-\frac{L_1}{2} < x < \frac{L_1}{2} \quad -\frac{L_1}{2} < y < \frac{L_1}{2} \quad L < z < L_1 \quad (7)$$

where $L_1 > L$ and the volume of the cavity is $V = L_1^2(L - L_1)$. The mode functions for this box have components

$$f_x(\mathbf{k}j, \mathbf{r}) = \sqrt{8}(\mathbf{e}_{\mathbf{k}j})_x \cos\left[k_x\left(x + \frac{L_1}{2}\right)\right] \sin\left[k_y\left(y + \frac{L_1}{2}\right)\right] \sin\left[k_z\left(z - L\right)\right]$$

$$f_y(\mathbf{k}j, \mathbf{r}) = \sqrt{8}(\mathbf{e}_{\mathbf{k}j})_y \sin\left[k_x\left(x + \frac{L_1}{2}\right)\right] \cos\left[k_y\left(y + \frac{L_1}{2}\right)\right] \sin\left[k_z\left(z - L\right)\right] \qquad (8)$$

$$f_z(\mathbf{k}j, \mathbf{r}) = \sqrt{8}(\mathbf{e}_{\mathbf{k}j})_z \sin\left[k_x\left(x + \frac{L_1}{2}\right)\right] \sin\left[k_y\left(y + \frac{L_1}{2}\right)\right] \cos\left[k_z\left(z - L\right)\right]$$

where $\mathbf{e}_{\mathbf{k}j}$ are polarization unit vectors and the allowed values of $\mathbf{k}$ have components

$$k_x = \frac{l\pi}{L_1}, \quad k_y = \frac{m\pi}{L_1}, \quad k_z = \frac{n\pi}{L_1 - L}, \quad l, m, n = 0, 1, \ldots \tag{9}$$

As mentioned before, at the end we take the limit $L, L_1 \to +\infty$.

Since we have replaced the classical stress tensor with a quantum operator we have to replace the difference on the RHS of (4) with a difference between quantum averages of the stress tensor operator. These averages must be calculated on quantum states reflecting the different physical situation at the two sides of the wall: the bare vacuum on its left side and the atom on the right side, which we assume located at $\mathbf{r}_A = (0, 0, D)$, with $D > L$. On this basis, we take the *bare* vacuum state of the electromagnetic field for the space on the left side of the wall, which we indicate with $|0\rangle$. As for the right side of the wall, we use the *dressed* vacuum state, that is the vacuum state corrected by the presence of the atom. We obtain this state at the lowest significant order in the atom‑field interaction. It is very convenient to describe the atom‑field interaction using an effective interaction Hamiltonian, valid both in the near and the far zone,

given by [15]

$$W = -\frac{1}{2} \sum_{\mathbf{k}\mathbf{k}'jj'} \alpha(k)\mathbf{E}_{\mathbf{k}j}(\mathbf{r}_A) \cdot \mathbf{E}_{\mathbf{k}'j'}(\mathbf{r}_A) \qquad (10)$$

where

$$\mathbf{E}_{\mathbf{k}j}(\mathbf{r}) = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}(a_{\mathbf{k}j} - a_{\mathbf{k}j}^\dagger)\mathbf{f}(\mathbf{k}j, \mathbf{r}) \qquad (11)$$

are the Fourier components of the electric field (5) and $\alpha(k)$ is the dynamical polarizability of the atom. Using first order perturbation theory with the interaction (10), we get the dressed vacuum state as

$$|\tilde{0}\rangle = |0\rangle + |1\rangle \qquad (12)$$

where

$$|1\rangle = -\frac{\pi}{V} \sum_{\mathbf{k}\mathbf{k}'jj'} \alpha(k)\frac{\sqrt{kk'}}{k + k'}\mathbf{f}(\mathbf{k}, j, \mathbf{r}_A)\cdot\mathbf{f}(\mathbf{k}', j', \mathbf{r}_A)|1_{\mathbf{k}j}1_{\mathbf{k}'j'}\rangle \qquad (13)$$

$1_{\mathbf{k}j}$ denoting the presence of a photon with wavevector $\mathbf{k}$ and polarization j.

Thus, the force density is expressed by

$$\sigma(x, y) = \langle\tilde{0}|S(x, y)|\tilde{0}\rangle - \langle0|S(x, y)|0\rangle \qquad (14)$$

where, for simplicity of notations,

$$S(x,y) = S_{zz}(x,y,L). \qquad (15)$$

The explicit expression of the operator $S(x,y)$ can be simply obtained from (2) in the following form

$$S(x,y) = -\frac{2\hbar c}{V} \sum_{\mathbf{k}\mathbf{k}'jj'} \left\{ A(\mathbf{k}j)A(\mathbf{k}'j')(a_{\mathbf{k}j} - a_{\mathbf{k}j}^\dagger)(a_{\mathbf{k}'j'} - a_{\mathbf{k}'j'}^\dagger) + \right.$$
$$\left. + B(\mathbf{k}j,\mathbf{k}'j')(a_{\mathbf{k}j} + a_{\mathbf{k}j}^\dagger)(a_{\mathbf{k}'j'} + a_{\mathbf{k}'j'}^\dagger) \right\}. \qquad (16)$$

where

$$A(\mathbf{k}j) = \sqrt{k}(\mathbf{e}_{\mathbf{k}j})_z \sin\left[k_x\left(x + \frac{L_1}{2}\right)\right] \sin\left[k_y\left(y + \frac{L_1}{2}\right)\right] \qquad (17)$$

and

$$B(\mathbf{k}j,\mathbf{k}'j') = \frac{1}{\sqrt{kk'}} \left\{ \left((\mathbf{e}_{\mathbf{k}j})_z k_x - (\mathbf{e}_{\mathbf{k}j})_x k_z\right)\left((\mathbf{e}_{\mathbf{k}'j'})_z k_x' - (\mathbf{e}_{\mathbf{k}'j'})_x k_z'\right) \cdot \right.$$
$$\cdot \cos\left[k_x\left(x + \frac{L_1}{2}\right)\right] \cos\left[k_x'\left(x + \frac{L_1}{2}\right)\right] \sin\left[k_y\left(y + \frac{L_1}{2}\right)\right] \sin\left[k_y'\left(y + \frac{L_1}{2}\right)\right] +$$
$$+ \left((\mathbf{e}_{\mathbf{k}j})_z k_y - (\mathbf{e}_{\mathbf{k}j})_y k_z\right)\left((\mathbf{e}_{\mathbf{k}'j'})_z k_y' - (\mathbf{e}_{\mathbf{k}'j'})_y k_z'\right) \cdot$$
$$\left. \cdot \sin\left[k_x\left(x + \frac{L_1}{2}\right)\right] \sin\left[k_x'\left(x + \frac{L_1}{2}\right)\right] \cos\left[k_y\left(y + \frac{L_1}{2}\right)\right] \cos\left[k_y'\left(y + \frac{L_1}{2}\right)\right] \right\}. \qquad (18)$$

Using eq. (12), from (14) we obtain at the first order in α,

$$\sigma(x,y) = 2\langle 0|S(x,y)|1\rangle = 2S_{01}(x,y). \qquad (19)$$

Using (13) and (16), we obtain

$$\sigma(x,y) = \frac{8\pi\hbar c}{V^2} \sum_{\mathbf{k}\mathbf{k}'jj'} \alpha(k)\frac{\sqrt{kk'}}{k+k'}\mathbf{f}(\mathbf{k}j,\mathbf{r}_A) \cdot \mathbf{f}(\mathbf{k}'j',\mathbf{r}_A) \times$$
$$\times \left[A(\mathbf{k}j)A(\mathbf{k}'j') + B(\mathbf{k}j,\mathbf{k}'j')\right]. \qquad (20)$$

Our system has a cylindrical symmetry around the axis

perpendicular to the wall and passing through the atom. Thus, being $\mathbf{P} = (0,0,L)$ the point common to this axis and the wall, the force density depends only on the distance ρ of a point of the plate from $\mathbf{P}$. In the far zone, where the dynamical polarizability $\alpha(k)$ is replaced with its static value, we obtain

$$\sigma(\rho) = \frac{\hbar c\alpha}{4\pi^3} \int_0^{+\infty} dx \left[I_1^2(x) + 2I_2^2(x) + I_3^2(x) + I_4^2(x)\right] \qquad (21)$$

where

$$I_1(x) = \int_0^{+\infty} dk\, k^3 e^{-kx} \int_0^\pi d\theta\, \sin^3\theta \cos(kd\cos\theta) J_0(k\rho\sin\theta)$$
$$I_2(x) = \int_0^{+\infty} dk\, k^3 e^{-kx} \int_0^\pi d\theta\, \sin\theta \cos\theta \sin(kd\cos\theta) J_0(k\rho\sin\theta)$$
$$I_3(x) = \int_0^{+\infty} dk\, k^3 e^{-kx} \int_0^\pi d\theta\, \sin^2\theta \cos(kd\cos\theta) J_1(k\rho\sin\theta)$$
$$I_4(x) = \int_0^{+\infty} dk\, k^3 e^{-kx} \int_0^\pi d\theta\, \sin^2\theta \cos\theta \sin(kd\cos\theta) J_1(k\rho\sin\theta)$$

$$(22)$$

where $J_\nu(x)$ is a Bessel function of the first kind of order ν [16]. Making use of known properties of Bessel functions [17], we finally obtain the following expression for the force density

$$\sigma(\rho) = \frac{\hbar c \alpha}{4\pi^2} \frac{17d^2 + 10\rho^2}{(d^2 + \rho^2)^{\frac{9}{2}}} \qquad (23)$$

where $d = D - L$ is the atom-wall distance. It is immediate to see that the force density is vanishing in both limits $d \to +\infty$ (atom infinitely distant from the wall) and $\rho \to +\infty$ (surface element of the wall infinitely distant from the atom).

III. THE INTEGRAL OF THE FORCE DENSITY AND NONADDITIVITY OF CASIMIR-POLDER FORCES

Once the force density (23) has been obtained, we can integrate it over the surface of the wall in order to obtain the total force $F_W(d)$ experienced by the wall. We easily obtain

$$F_W(d) = 2\pi \int_0^{+\infty} d\rho\, \rho\, \sigma(\rho) = \frac{3\hbar c \alpha}{2\pi d^5} \qquad (24)$$

which is the opposite of the force $F_A(d)$ acting on the atom given by equation (1).

This result may appear contradicting the well-established fact that Casimir-Polder forces are not additive [3, 18, 19]. For example, in the case of three atoms it is known that the force on one of them is not simply the sum of the forces due to the other two atoms separately. The system we are considering, and in particular the fact that we are focusing our attention on elements of the plate, raises a similar problem. Suppose we have the atom in front of a single element of the plate and that we calculate the force acting on this surface element. This force, obviously, will depend on the coordinate ρ of the element. Due to the nonadditivity of Casimir-forces, we expect that its integral over the entire plate should *not* give the total force, since the force on the wall should also contain three-body components involving the atom and two different elements of the wall (all these components are proportional to α). The solution of this seemingly contradictory point is related to the use of the stress-tensor method. The stress tensor operator (2) contains in its very expression the electric and magnetic field operators. These field operators, given by equations (5) and (6), involve the mode function $\mathbf{f}(\mathbf{k}j, \mathbf{r})$, which are taking into account the presence of the entire plate. As a consequence, we claim that the use of the operator $S(x, y)$

to calculate the force density on a surface element of the wall answers the following question: if we have a neutral atom in front of a conducting wall, what is the force acting on a surface element of the wall in the presence of the *entire* plate? On the basis of this consideration, the integral of such contribution over the wall surface must give the correct value for the total force, as we have explicitly shown. Hence, all many-body contributions to the atom-surface element Casimir-Polder force which are proportional to α are already included in our result (23). Moreover, we wish to stress that from the density of the force we can obtain much more information on the effects of the atom-wall Casimir-Polder interaction, such as torques or stresses on the wall due to the presence of the atom. These effects may be relevant in the recently proposed technological application of Casimir forces [21].

IV. CONCLUSIONS

In this paper we have considered the Casimir-Polder interaction between a neutral atom and a neutral conducting wall. We have calculated, using the quantum operator associated with the classical electromagnetic stress tensor, the force density on the plate, that is the force acting on each surface element of the wall. We have shown that the integral of this density gives the correct result for the total forced acting on the plate, i.e. the opposite of the force on the atom. This shows that our method based on the stress tensor, usually used for macroscopic bodies, enables to include easily, at the order considered, all nonadditive components of Casimir-Polder forces between the atom and two or more plate elements. In fact, the force calculated for each surface element is indeed the force acting on it in the presence of the entire plate. This happens because the stress tensor in its very definition contains the modes of the electromagnetic field, which take into account the presence of all macroscopic bodies in the system.

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